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HAUSDORFF DIMENSION OF ATTRACTORS FOR TWO DIMENSIONAL LORENZ TRANSFORMATIONS

BY

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ABSTRACT

A class of transformations on $[0,1]^2$, which includes transformations obtained by a Poincare section of the Lorenz equation, is considered. We prove that the Hausdorff dimension of the attractor of these transformations equals $z+1$ where z is the unique zero of a certain pressure function. Furthermore we prove that all vertical intersections with this attractor, except of countable many, have Hausdorff dimension z.

1. Introduction

One of the best known differential equations which exhibit chaotic behavior is the Lorenz equation (see [3]). Using a Poincare section one can reduce it to a discrete time dynamical system on $[0,1]^2$. This dynamical system is qualitatively the same as the transformations given by $F(x,y) = (T(x), g(x,y))$ where $g(x,.)$: $[0,1] \to (0,1)$ is a contraction for each $x \in [0,1]$ and T: $[0,1] \to [0,1]$ is piecewise strictly increasing with two monotone pieces.

We use a slightly more general definition. A transformation F: $[0,1]^2 \to [0,1]^2$ is called a two dimensional Lorenz transformation if $F(x,y) = (T(x), g(x,y))$ where:

(1) T: $[0,1] \to [0,1]$ is piecewise monotonic. This means, that there are $c_i \in [0,1]$ for $0 \le i \le N$ with $0 = c_0 < \ldots < c_N = 1$ such that $T|_{(c_i,c_{i+1})}$ is continuous and strictly monotone for $0 \le i < N$.

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(2) $g\colon [0,1]^2 \to (0,1)$ is C^1 on $P \times [0,1]$, where $P = [0,1] \setminus \bigcup_{0\le i\le N} c_i$. Furthermore we suppose $\sup|\partial g/\partial x| < \infty$, $\sup|\partial g/\partial y| := k < 1$ and $\inf|\partial g/\partial y| > 0$.

(3) $F((c_i, c_{i+1}) \times [0,1]) \cap F((c_j, c_{j+1}) \times [0,1]) = \emptyset$ for distinct i,j with $0 \le i,j < N$.

Denote the projection to the x-axis by Π and set $X = \bigcap_{n=-\infty}^{\infty} F^n(\Pi^{-1}P)$. Then X is the attractor for all orbits under F, which start in a point $\mathbf{x} \in \Pi^{-1}P$ and such that $F^n\mathbf{x} \in \Pi^{-1}P$ for all $n \ge 0$. Observe that F is invertible on X. For $a \in [0,1]$ and a subset A of $[0,1]^2$ denote the vertical intersection $A \cap \Pi^{-1}a$ by A_a and the Hausdorff dimension of A by $HD(A)$. The aim of this paper is to prove that the Hausdorff dimension $HD(X_\sigma)$ of a vertical intersection X_σ for $\sigma \in [0,1]$ equals the unique zero z of the function $t \mapsto p(F, t\varphi)$ with $\varphi = \log|\partial g/\partial y|$. Furthermore we will show $HD(X) = z + 1$.

Such results are known for one dimensional dynamical systems, for axiom A diffeomorphisms and for self affine sets. In [12] it is shown that $HD(X)$ equals the unique zero of $t \mapsto p(T|_X, -t\log|T'|)$ if T is an expanding piecewise monotonic transformation on $[0,1]$ and X is a completely invariant subset of $[0,1]$. This is also proved in [5] with slightly more general assumptions. In [10] one finds the same formula as we prove in Theorem 1, but for an axiom A diffeomorphism. In [13] a non-invertible map semiconjugate with a finite shift is considered and the same formula as in Theorem 2 is obtained. For the self affine case see the books of Falkoner ([1], [2]) and Pesin ([11]).

Remark 1: The points $c_i \in [0,1]$ for $0 \le i \le N$ are called critical points. Set $C = \{c_i\colon 1 \le i \le n\}$. For $x \in [0,1)$ set $T_+^n(x) = \lim_{y\searrow x} T^n(y)$ and for $x \in (0,1]$ set $T_-^n(x) = \lim_{y\nearrow x} T^n(y)$.

We can always declare finitely many additional points to be critical to get the following property of C:

(1) If $T_a^k(c_i) = T_b^l(c_j)$ and $T_a(c_i), T_b(c_j) \notin C$ for some $a,b \in \{+,-\}$, $k,l \ge 0$ and $c_i, c_j \in C$ then either $\min\{k,l\} = 0$ or $T_a^{k-r}(c_i) = T_b^{l-r}(c_j)$ for $1 \le r \le \min\{k,l\}$.

(2) If $c_i = T_a^l(c_j)$ and $T_a(c_j) \notin C$ then $l = 0$ and $c_i = c_j$.

It is shown in [8] that these properties can be achieved by declaring finitely many additional points to be critical. We need this property to prove Lemma 7 below.

Set $\mathcal{V} = \{(c_i, c_{i+1})\colon 0 \le i < N\}$ and $\mathcal{Z} = \Pi^{-1}\mathcal{V}$. In our situation the following definition of pressure will be useful. Let f be a bounded and piecewise continuous function, that means $f|_Z$ is continuous for $Z \in \mathcal{Z}$. Set $\mathcal{Z}_n = \bigvee_{i=0}^{n} F^{-i}\mathcal{Z}$ and

$S_nf(\mathbf{x}) = \sum_{i=0}^{n-1} f(F^i\mathbf{x})$. Then we define $p_n(F,f) = \sum_{Z\in\mathcal{Z}_n} \sup_{\mathbf{x}\in Z} e^{S_nf(\mathbf{x})}$ and $p(F,f) = \lim_{n\to\infty} \frac{1}{n}\log p_n(F,f)$. The limit in the above definition exists, since $p_{n+k}(F,f) \le p_n(F,f)p_k(F,f)$ holds (see [16]) and since f is bounded. Set $E = \bigcap_{n=0}^{\infty} T^{-i}P$. This is the set of points $x \in [0,1]$, for which all iterates of T stay in P. The exact results are:

THEOREM 1: *Let F be a two dimensional Lorenz transformation. Suppose T is topologically transitive and $\mathcal{V}$ is a generator for T. If the family of maps $y \mapsto \partial g/\partial y(x,y)$ is uniformly Hölder continuous, then $t \mapsto p(F, t\varphi)$ has a unique zero z and $HD(X_\sigma) = z$ holds for all $\sigma \in E$.*

We say the transformation T: $[0,1] \to [0,1]$ is expanding if $T|_V$ is C^1 for $V \in \mathcal{V}$ and $\inf |T'| > 1$.

THEOREM 2: *If the same assumptions as in Theorem 1 hold and if additionally T is expanding, then $HD(X) = z + 1$.*

2. The upper bound for $HD(X_\sigma)$

We begin with the proof of the desired properties of the pressure function.

LEMMA 1: *Suppose F is a two dimensional Lorenz transformation. Then the map $t \mapsto p(F, t\varphi)$ is continuous, strictly decreasing and tends to $-\infty$ for $t \to \infty$. Hence there is a unique zero $z \ge 0$.*

Proof: Set $\gamma = -\sup_{\mathbf{x}\in\Pi^{-1}P} \varphi(\mathbf{x})$. Then $\gamma > 0$ since $\sup |\partial g/\partial y| < 1$ holds. For $0 \le t < s$ we get $s\varphi = t\varphi + (s-t)\varphi \le t\varphi - (s-t)\gamma$. With the same proof as in [16] this implies

$$p(F, s\varphi) \le p(F, t\varphi) - (s-t)\gamma.$$

This inequality proves that $t \mapsto p(F, t\varphi)$ is strictly decreasing. Since

$$0 \le p(F,0) = \lim_{n\to\infty} \frac{1}{n} \log card(\mathcal{Z}_n) \le card(\mathcal{Z}) < \infty$$

holds, we also get $\lim_{t\to\infty} p(F, t\varphi) = -\infty$.

The continuity of the map $t \mapsto p(F, t\varphi)$ follows immediatly from the well-known formula $|p(F,f) - p(F,g)| \le ||f-g||_\infty$ for bounded functions f and g (see [16]). ∎

We already can tackle the proof that the unique zero z of $t \mapsto p(F, t\varphi)$ is an upper bound for $HD(X_\sigma)$ for $\sigma \in E$. Similar arguments are given in [10] and [12]. For a subset $A \subseteq [0,1]^2$ we denote the diameter of A by $|A|$. Denote the set of all finite or countable covers of A by intervals with diameter less than ϑ by $\mathcal{U}(A,\vartheta)$ and set $\nu_{t,\vartheta}(A) = \inf_{\mathcal{C}\in\mathcal{U}(A,\vartheta)} \sum_{C\in\mathcal{C}} |C|^t$. Then $\nu_t(A) = \lim_{\vartheta\to 0} \nu_{t,\vartheta}(A)$ is called t-dimensional Hausdorff measure, where the limit exists since $\nu_{t,\vartheta}$ is increasing when ϑ is decreasing but might be ∞. Set $\mathcal{I}_n = F^n\mathcal{Z}_n$ for $n \geq 0$ and observe that $X \subseteq \bigcap_{n\geq 0} \bigcup_{I\in\mathcal{I}_n} I$ holds.

LEMMA 2: *Let F be a two dimensional Lorenz transformation and choose $\sigma \in E$. Then $HD(X_\sigma) \leq z$.*

Proof: For $n \geq 0$ choose $I \in \mathcal{I}_n$ with $\sigma \in \Pi I$. Then $Z_I := F^{-n}I$ is an element of $\mathcal{Z}_n$. Since T^n is invertible from $V_I := \Pi Z_I \to \Pi I$ there is a unique $\eta_I \in V_I$ with $T^n\eta_I = \sigma$. From the mean value theorem and the chain rule we obtain the existence of a $\zeta \in [0,1]$ with

$$|I_\sigma| = |[0,1]| \left|\frac{\partial}{\partial y} F^n(\eta_I,\zeta)\right| = \prod_{i=0}^{n-1} \left|\frac{\partial g}{\partial y}(F^i(\eta_I,\zeta))\right|.$$

Fix an arbitrary $\vartheta \in (0,1)$. Then we find an n, such that $|I_\sigma| \leq \vartheta$ for all $I \in \mathcal{I}_n$ since $\sup|\partial g/\partial y| = k < 1$ holds.

Choose $t > z$. It is clear that $\bigcup_{I\in\mathcal{I}_n} I_\sigma$ covers X_σ. Hence

$$\begin{aligned}
\nu_{t,\vartheta}(X_\sigma) &\leq \sum_{\substack{I\in\mathcal{I}_n\\ \sigma\in\Pi I}} |I_\sigma|^t \leq \sum_{\substack{I\in\mathcal{I}_n\\ \sigma\in\Pi I}} \sup_{y\in[0,1]} \exp\left(t\sum_{i=0}^{n-1} \log\left|\partial g/\partial y(F^i(\eta_I,y))\right|\right)\\
&\leq \sum_{\substack{I\in\mathcal{I}_n\\ \sigma\in\Pi I}} \sup_{x\in V_I} \sup_{y\in[0,1]} \exp\left(t\sum_{i=0}^{n-1} \log\left|\partial g/\partial y(F^i(x,y))\right|\right)\\
&\leq \sum_{Z\in\mathcal{Z}_n} \sup_{\mathbf{x}\in Z} e^{tS_n\varphi(\mathbf{x})} = p_n(F,t\varphi).
\end{aligned}$$

To show $HD(X_\sigma) \leq t$ it suffices to show that $\nu_t(X_\sigma)$ is bounded. We know that $\frac{1}{n}\log p_n(F,t\varphi)$ converges to $p(F,t\varphi)$, where $p(F,t\varphi) < 0$ holds because of the choice of $t > z$. Hence $p_n(F,t\varphi)$ tends to zero and is bounded. We conclude that $\nu_{t,\vartheta}(X_\sigma)$ is bounded for all $\vartheta > 0$. Hence $\nu_t(X_\sigma)$ is bounded for all $t > z$ and $HD(X_\sigma) \leq z$ follows. ∎

Observe that the definition of F implies $\Pi\mathcal{Z}_n = \mathcal{V}_n$, where $\mathcal{V}_n$ is defined by $\mathcal{V}_n = \bigvee_{i=0}^n T^{-i}\mathcal{V}$. Hence $h_{top}(T) = p(F,0)$ follows, where $h_{top}(T)$ denotes the topological entropy of the one dimensional dynamical system $([0,1],T)$.

Remark 2: If $z = 0$ then $HD(X_\sigma) = z$ is already shown. Hence we will assume $z > 0$ during the rest of this paper. With Lemma 1 this implies $h_{top}(T) > 0$.

3. The lower bound for $HD(X_\sigma)$

We fix a $t \in (0, z)$ and set $\psi = t\varphi$. Then $p(F, \psi) > 0$ and we can choose an $\varepsilon > 0$ such that $p(F, \psi) - 2\varepsilon > 0$ holds. We now use also the two sided partition $\mathcal{Z}_{k,l} = \bigvee_{i=-l}^{k} F^i \mathcal{Z}$. All three families of sets $\mathcal{Z}_k$, $\mathcal{I}_l$ and $\mathcal{Z}_{k,l}$ are partitions of X and for $\mathbf{x} \in X$ we denote the unique element of the partition which includes $\mathbf{x}$ by $Z_k(\mathbf{x})$, $I_l(\mathbf{x})$ and $Z_{k,l}(\mathbf{x})$.

LEMMA 3: *Suppose $\mathcal{V}$ is a one sided generator for T. Then $\mathcal{Z}$ is a two sided generator for F on X. Furthermore for all $\delta > 0$ there is an $n(\delta)$ such that $|V| \leq \delta$ for all $V \in \mathcal{V}_{n(\delta)}$ and $|Z| \leq \delta$ for all $Z \in \mathcal{Z}_{n(\delta),n(\delta)}$ holds.*

Proof: To prove the first assertion choose $\mathbf{x} \neq \mathbf{y} \in X$. If $\Pi\mathbf{x} \neq \Pi\mathbf{y}$ the assertion is trivial. If $\Pi\mathbf{x} = \Pi\mathbf{y}$ there is an $n \geq 0$ with $I_n(\mathbf{x}) \cap I_n(\mathbf{y}) = \emptyset$ since g is a contraction on every vertical line and it follows that $\mathcal{Z}$ is a generator.

The proof of the second assertion is standard. ∎

LEMMA 4: *Suppose the family of maps $y \mapsto \frac{\partial g}{\partial y}(x, y)$ is uniformly Hölder continuous and $\mathcal{V}$ is a generator. Then there is a function $\hat{\psi}$: $[0,1]^2 \to \mathbb{R}$ and an $N \in \mathbb{N}$ such that $\hat{\psi}|_Z$ is constant for $Z \in \mathcal{Z}_N$ and $p(F, \hat{\psi}) \leq p(F, \psi) \leq p(F, \hat{\psi}) + \varepsilon$.*

Proof: Since the family of maps $y \mapsto \frac{\partial g}{\partial y}(x, y)$ is uniformly Hölder continuous and $|\partial g / \partial y|$ is bounded away from 0 and ∞ also the maps $y \mapsto \psi(x, y)$ are uniformly Hölder continuous for each $x \in P$. Set $s(\mathbf{x}) = (\Pi\mathbf{x}, 0)$ for $\mathbf{x} \in \Pi^{-1}P$ and

$$u_n(\mathbf{x}) = \sum_{j=0}^{n} \psi(F^j \mathbf{x}) - \psi(F^j s(\mathbf{x})).$$

Since $|F^j(\mathbf{x}) - F^j(s(\mathbf{x}))| \leq k^j |\mathbf{x} - s(\mathbf{x})| \leq k^j$ holds, we get $|u_n(\mathbf{x}) - u_m(\mathbf{x})| \leq c\sum_{j=m+1}^{n} (k^\alpha)^j$ for a $c > 0$ and an $\alpha > 0$. Hence $u(\mathbf{x}) = \lim_{n\to\infty} u_n(\mathbf{x})$ exists and is continuous on Z for $Z \in \mathcal{Z}$ as a uniform limit of continuous functions. Furthermore u is bounded.

Set $\tilde{\psi} = \psi - u + u \circ F$. Then $p(F, \tilde{\psi}) = p(F, \psi)$ follows by the definition of pressure and we compute

$$\begin{aligned}\tilde{\psi}(\mathbf{x}) =& \psi(\mathbf{x}) - \sum_{j=0}^{\infty} \psi(F^j \mathbf{x}) - \psi(F^j s(\mathbf{x})) \\ &+ \sum_{j=0}^{\infty} \psi(F^{j+1} \mathbf{x}) - \psi(F^j s(F\mathbf{x})) \\ =& \psi(s(\mathbf{x})) - \sum_{j=0}^{\infty} \psi(F^{j+1} s(\mathbf{x})) - \psi(F^j s(F\mathbf{x})).\end{aligned}$$

Since $s(F\mathbf{x}) = s(Fs(\mathbf{x}))$ the above equation proves that $\tilde{\psi}$ is constant on vertical lines. Furthermore $\tilde{\psi}$ is continuous on the elements of $\mathcal{Z}$ and bounded. Hence we find with use of Lemma 3 an $N \in \mathbb{N}$ such that $||\tilde{\psi} - \sum_{Z \in \mathcal{Z}_N} \chi_Z \inf_{\mathbf{x} \in Z} \tilde{\psi}(\mathbf{x})||_\infty < \varepsilon$. Set $\hat{\psi} = \sum_{Z \in \mathcal{Z}_N} \chi_Z \inf_{\mathbf{x} \in Z} \tilde{\psi}(\mathbf{x})$; then the two assertions of the lemma are fulfilled. ∎

Set $\mathcal{I}_\infty = \{\bigcap_{n \geq 0} I_n \neq \emptyset \colon I_n \in \mathcal{I}_n\}$ and for $\sigma \in P$ set

$$\mathcal{I}_\sigma = \{I \colon I \in \mathcal{I}_n \text{ for some } n \geq 0 \text{ and } \sigma \in \Pi I\}$$

and

$$\mathcal{I}_{\infty,\sigma} = \{I \in \mathcal{I}_\infty \colon \sigma \in \Pi I\}.$$

We will use the function $\hat{\psi}$ constructed above to find an ergodic F-invariant measure μ on X with $\mu(\bigcup_{I \in \mathcal{I}_{\infty,\sigma}} I) > 0$ for all $\sigma \in E$ and $h_\mu(F) + t \int \varphi \, d\mu > 0$. For the construction of this measure we have to introduce the Markov diagram over $\mathcal{W}$ of the one dimensional dynamical system $([0,1], T)$, where $\mathcal{W}$ is a finite partition refining $\mathcal{V}$. This concept is due to Hofbauer (see [6] and [7]).

For an open subinterval D of a $Z_0 \in \mathcal{W}$ the nonempty intervals among $T(D) \cap Z$ for $Z \in \mathcal{W}$ are called successors of D. We write $D \to C$ if C is a successor of D. All successors of a D are again open subintervals of a $Z_1 \in \mathcal{W}$ since T is piecewise monotonic. Hence building successors can be iterated.

Set $\mathcal{D}_0 = \mathcal{W}$ and define $\mathcal{D}_i = \mathcal{D}_{i-1} \cup \{D \colon \exists C \in \mathcal{D}_{i-1} \text{ with } C \to D\}$. Then all the $\mathcal{D}_i$ are finite sets since $card(\mathcal{W}) < \infty$ and each $D \in \mathcal{D}_{i-1}$ has not more than $card(\mathcal{W})$ successors. Hence $\mathcal{D} = \bigcup_{i=0}^\infty \mathcal{D}_i$ is at most countable and we call the oriented graph $(\mathcal{D}, \to)$ with arrows $C \to D$, if D is a successor of C, the Markov diagram of $([0,1], T)$ over $\mathcal{W}$.

A finite or infinite sequence $D_0 D_1 \ldots$ with $D_i \in \mathcal{D}$ is called a path if $D_{i-1} \to D_i$ holds for $i > 0$. A subset $\mathcal{C} \subseteq \mathcal{D}$ is called closed, if $D \in \mathcal{C}$ and $D \to C$ imply $C \in \mathcal{C}$. It is called irreducible if, whenever $C \in \mathcal{C}$ and $D \in \mathcal{C}$, a path leads from D to C and no subset of $\mathcal{D}$ which contains $\mathcal{C}$ has this property.

LEMMA 5: *Suppose that T is topologically transitive, $h_{top}(T) > 0$ and $\mathcal{V}$ is a generator for T. Then there is an irreducible closed subset $\mathcal{C} \subseteq \mathcal{D}$, such that all irreducible subsets of $\mathcal{D} \setminus \mathcal{C}$ consist only of a single closed path and for all $D \in \mathcal{D} \setminus \mathcal{C}$ there is a path starting in D, which leads to $\mathcal{C}$. Furthermore there is a finite subset $\mathcal{A} \subseteq \mathcal{C}$ with $\bigcup_{D \in \mathcal{A}} \overline{D} = [0,1]$.*

Proof: Since $\mathcal{V}$ is a generator and $\mathcal{W}$ refines $\mathcal{V}$ also $\mathcal{W}$ is a generator. This, and the first and the second assumption of the lemma are used to prove the first part of the result. See [4] for a detailed proof.

For the second part we use Theorem 10 in [6]. It states that there is a finite set $\mathcal{A} \subseteq \mathcal{C}$ with $\bigcup_{D\in\mathcal{A}} D = \bigcup_{D\in\mathcal{C}} D$. The two infinite sets appearing in this theorem are empty, since $\mathcal{W}$ is a generator. Since $\mathcal{C}$ is closed and T is topologically transitive we get $\bigcup_{D\in\mathcal{C}} \overline{D} = [0,1]$. ∎

Remember that the function $\hat{\psi}$ constructed in Lemma 4 is constant on the elements of $\mathcal{Z}_N$. We can conceive $\hat{\psi}$ as a function in one variable, which then is constant on $\mathcal{W} := \Pi\mathcal{Z}_N$. For $D \in \mathcal{D}$ let $\hat{\psi}_D$ be the unique number with $\hat{\psi}(x) = \hat{\psi}_D$ for all $x \in D$. We define a $\mathcal{D}\times\mathcal{D}$-matrix $\mathbf{M}$ by

$$\mathbf{M}_{CD} = \begin{cases} e^{\hat{\psi}_C} & \text{if } C \to D, \\ 0 & \text{otherwise.} \end{cases}$$

Since the number of successors of a $C \in \mathcal{D}$ is bounded by $card(\mathcal{W})$ we get $\sum_{D\in\mathcal{D}} \mathbf{M}_{CD} \le K$ with $K = card(\mathcal{W})||e^{\hat{\psi}}||_\infty < \infty$.

For $\mathbf{v} \in l^\infty(\mathcal{D})$ and $n > 0$ we compute

$$||\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}\mathbf{v}_D||_\infty = \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}\mathbf{v}_D| \le ||\mathbf{v}||_\infty \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}|$$

and choosing $\mathbf{v} = (1,1,\ldots)$ we get $||\mathbf{M}^n||_\infty = \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}|$. Similarly we compute for $\mathbf{u} \in l^1(\mathcal{D})$

$$||\sum_{C\in\mathcal{D}} \mathbf{u}_C\mathbf{M}^n{}_{CD}||_1 \le \sum_{C\in\mathcal{D}} |\mathbf{u}_C|\,|\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}| \le ||\mathbf{u}||_1 \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}|.$$

To see $||\mathbf{M}^n||_1 = \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}|$ one has to choose an appropriate sequence $\mathbf{u}_i$ of $l^1(\mathcal{D})$-vectors which have one entry 1 and all others 0 such that $\sup ||\mathbf{u}_i\mathbf{M}^n||_1 = \sup_{C\in\mathcal{D}} |\sum_{D\in\mathcal{D}} \mathbf{M}^n{}_{CD}|$ holds.

Hence $\mathbf{u} \to \mathbf{uM}$ is an $l^1(\mathcal{D})$-operator and $\mathbf{v} \to \mathbf{Mv}$ is an $l^\infty(\mathcal{D})$-operator. Both operators have the same norm, denoted by $||\mathbf{M}||$. They also have the same spectral radius denoted by $r(\mathbf{M})$ since $r(\mathbf{M}) = \limsup_{n\to\infty} \sqrt[n]{||\mathbf{M}^n||}$. Furthermore $\log r(\mathbf{M}) = p(F, \hat{\psi})$ holds. This is shown in the same way as Lemma 6 (1) in [12]. Let $\mathcal{C} \subseteq \mathcal{D}$ be the closed and irreducible subset found in Lemma 5.

Lemma 6: *Suppose* $r(\mathbf{M}) > \limsup_{n\to\infty} \sqrt[n]{||e^{S_n\hat{\psi}}||}$. *Then there are nonnegative and nonzero* $\mathbf{v} \in l^\infty(\mathcal{D})$ *and* $\mathbf{u} \in l^1(\mathcal{D})$ *with* $\mathbf{Mv} = r(\mathbf{M})\mathbf{v}$ *and* $\mathbf{uM} = r(\mathbf{M})\mathbf{u}$ *and* $\mathbf{u}_D > 0$ *and* $\mathbf{v}_D > 0$ *for all* $D \in \mathcal{C}$ *where* $\mathcal{C}$ *denotes the irreducible closed subset of* $\mathcal{D}$ *of Lemma 5.*

Proof: Set $\mathbf{M}_n = \mathbf{M}|_{\mathcal{D}\setminus\mathcal{D}_n}$. Lemma 8 in [8] gives $\limsup_{n\to\infty} r(\mathbf{M}_n) \le \limsup_{n\to\infty} \sqrt[n]{||e^{S_n\hat{\psi}}||}$ and we can find an $n \ge 0$ with $r(\mathbf{M}) > r(\mathbf{M}_n)$. Now the same proof as that of Corollary 1(ii) to Theorem 9 in [6] shows the existence of nonzero and nonnegative $\mathbf{v} \in l^\infty(\mathcal{D})$ and $\mathbf{u} \in l^1(\mathcal{D})$ with $\mathbf{Mv} = r(\mathbf{M})\mathbf{v}$ and $\mathbf{uM} = r(\mathbf{M})\mathbf{u}$. Observe that $\mathbf{u}_A > 0$ implies $\mathbf{u}_B > 0$ for all $B \in \mathcal{D}$ which are met by a path in $\mathcal{D}$ starting in A. Similarly $\mathbf{v}_A > 0$ implies $\mathbf{v}_B > 0$ for $B \in \mathcal{D}$ whenever there is a path starting in B which meets A. Choose $C \in \mathcal{D}$ with $\mathbf{u}_C > 0$. Since there is a path from C to every element D of $\mathcal{C}$ we get $\mathbf{u}_D > 0$ for all $D \in \mathcal{C}$. If $\mathbf{v}_C > 0$ for a $C \in \mathcal{C}$ then $\mathbf{v}_D > 0$ for all $D \in \mathcal{C}$ follows and the Lemma is proved. Suppose $\mathbf{v}_D = 0$ for all $D \in \mathcal{C}$. In $\mathcal{D} \setminus \mathcal{C}$ all irreducible subsets consist only of a single closed path. Denote the set of these single closed paths by $\mathcal{S}$. If $\mathbf{v}_D = 0$ for all $D \in \mathcal{C}$ this implies

$$\begin{aligned} r(\mathbf{M}) &\leq \limsup_{n\to\infty} \left(\sup\{\prod_{i=1}^{n-1} \mathbf{M}_{C_i C_{i+1}} \colon C_1 \to \cdots \to C_n \text{ is in an element of } \mathcal{S}\} \right)^{\frac{1}{n}} \\ &\leq \limsup_{n\to\infty} \sqrt[n]{\|e^{S_n\hat{\psi}}\|}. \end{aligned}$$

The last inequality holds since for every $n \geq 0$ and every path $C_1 \to \cdots \to C_n$ of length n there is an $x \in P$ with $T^i(x) \in C_i$ for $1 \leq i \leq n$. ∎

Set $\mathcal{G} = \{\ldots D_{-1}D_0D_1\ldots \colon D_k \in \mathcal{D} \text{ and } D_k \to D_{k+1} \text{ for } k \in \mathbb{Z}\}$ and denote the shift transformation on $\mathcal{G}$ by S. Then S is bijective on $\mathcal{G}$. Set $[C_{-k}\ldots C_n]_n = \{\ldots D_{-1}D_0D_1\ldots \in \mathcal{G} \colon D_i = C_i \text{ for } -k \leq i \leq n\}$. Such sets are also written as ${}_{-k}[C_{-k}\ldots C_n]$. Furthermore we define ${}_{-k}[C_{-k}\ldots] = \bigcap_{n\geq 0}({}_{-k}[C_{-k}\ldots C_n])$.

Lemma 7: *Suppose $\ldots \to C_{-1} \to C_0$ and $\ldots \to D_{-1} \to D_0$ are both infinite inverse paths in $\mathcal{D}$ with the property that C_i and D_i are subintervals of the same $Z_i \in \mathcal{W}$ for all $i \leq 0$. Then $C_i = D_i$ holds for all $i \leq 0$.*

Proof: Set $D_0 = (a, b)$ and suppose without loss of generality (see Lemma 1 in [8]) that $a = T^{n_0}(c_i)$ and $b = T^{n_1}(c_j)$ for $c_i, c_j \in C$ and $n_0 \geq n_1$. Lemma 1 in [14] gives $\bigcap_{i=0}^{l} T^{-i}D_{-l+i} = D_{-l} \cap \bigcap_{i=1}^{l} T^{-i}Z_{-l+i}$ where Z_{-j} is the unique element of $\mathcal{W}$ which contains D_{-j}. For $l = n_0$ the sets $\bigcap_{i=0}^{n_0} T^{-i}D_{-n_0+i}$ and $\bigcap_{i=0}^{n_0} T^{-i}Z_{-n_0+i}$ have that endpoint in common, which is mapped to a and this property holds for $l \geq n_0$ since the D_{-j} for $n_0 \leq j \leq l$ are elements of a path in $\mathcal{D}$. If $l \geq n_1$ we get that $\bigcap_{i=0}^{l} T^{-i}D_{-l+i}$ and $\bigcap_{i=0}^{l} T^{-i}Z_{-l+i}$ have that endpoint in common, which is mapped to b. Here we use Remark 1. Hence we get for $l \geq \max\{n_0, n_1\}$

$$\bigcap_{i=0}^{l} T^{-i}D_{-l+i} = \bigcap_{i=0}^{l} T^{-i}Z_{-l+i}.$$

The same argument gives the existence of an $m \in \mathbb{N}$ such that for $k \geq m$ we get $\bigcap_{i=0}^{l} T^{-i}C_{-l+i} = \bigcap_{i=0}^{l} T^{-i}Z_{-l+i}$ where again Z_{-j} is the unique element of $\mathcal{W}$ which contains D_{-j} for $0 \leq j \leq l$. Set $N = \max\{n, m\}$. Since D_{-j} and C_{-j} are contained in the same $Z_{-j} \in \mathcal{W}$ and with Lemma 1 of [6] we get

$$D_0 = T^N \bigcap_{i=0}^{l} T^{-i}D_{-l+i} = T^N \bigcap_{i=0}^{l} T^{-i}Z_{-l+i} = T^N \bigcap_{i=0}^{l} T^{-i}C_{-l+i} = C_0.$$

Since this argument can be applied on every C_{-i} and D_{-i} for $i \geq 0$ the lemma is proved. ∎

LEMMA 8: *Suppose $\mathcal{V}$ is a generator. Then there is an injective map $\phi\colon \mathcal{G} \setminus \mathcal{N} \to X$ where $\mathcal{N}$ is S-invariant and a countable union of sets of the form ${}_m[D_m \ldots]$ for an $m \in \mathbb{Z}$. Furthermore $F\phi = \phi S$ holds and for $\sigma \in D_0$ we have $\phi[\ldots D_{-1}D_0]_0 \subseteq \bigcup_{I \in \mathcal{I}_{\infty,\sigma}} I$.*

Proof: For each $D \in \mathcal{D}$ we define a set $\mathcal{P}(D)$ containing two paths as follows:

Set $D = (a, b)$. There is a unique path $D_0D_1\ldots$ with $D_0 = D$ such that $T^i(a)$ is an endpoint of D_i for all $i \geq 0$ and there is a unique path $C_0C_1\ldots$ with $C_0 = D$ such that $T^i(b)$ is an endpoint of C_i for all $i \geq 0$. Set $\mathcal{P}(D) = \{D_0D_1\ldots, C_0C_1\ldots\}$.

For $\ldots D_{-1}D_0D_1\ldots \in \mathcal{G}$ set $B_{0,0} = \Pi^{-1}D_0$ and $B_{k,l} = \bigcap_{i=-l}^{k} F^{-i}\Pi^{-1}D_i$. Then $B_{k,l} = F^l\Pi^{-1}\bigcap_{i=-l}^{k} T^{-(i+l)}D_i$ holds and since $D_{-l}\ldots D_k$ is a path in $\mathcal{D}$ we get $B_{k,l} \neq \emptyset$.

Furthermore we get either $\bigcap_{k\geq 0} B_{k,0} = \bigcap_{k\geq 0} \overline{B_{k,0}} = \Pi^{-1}x$ for an $x \in D_0$, or $D_mD_{m+1}\ldots \in \mathcal{P}(D_m)$ for some $m \geq 0$. Since $\mathcal{V}$ is a generator this $x \in D_0$ is unique. Set

$$\mathcal{N} = \{\ldots D_{-1}D_0D_1\ldots \in \mathcal{G}\colon \exists m \in \mathbb{Z} \text{ with } D_mD_{m+1}\ldots \in \mathcal{P}(D_m)\}.$$

Then $\mathcal{N}$ is S-invariant and a countable union of sets of the form ${}_m[D_m\ldots]$. We have $\Pi B_{0,l} = \Pi\bigcap_{i=-l}^{0} F^{-i}\Pi^{-1}D_i = \bigcap_{i=-l}^{0} T^{-i}D_i = D_0$ and $B_{0,l+1} \subseteq B_{0,l}$ for all $l \geq 0$. Hence we can define $\phi\colon \mathcal{G}\setminus\mathcal{N} \to X$ by $\phi(\ldots D_{-1}D_0D_1\ldots) = \bigcap_{l=0}^{\infty} B_{0,l} \cap \bigcap_{k=0}^{\infty} B_{k,0}$.

Suppose now $\phi(\ldots C_{-1}C_0C_1\ldots) = \phi(\ldots D_{-1}D_0D_1\ldots)$ holds for two different elements of $\mathcal{G}\setminus\mathcal{N}$. Then the assumptions of Lemma 7 are fulfilled and we get $D_i = C_i$ for $i \leq 0$. But now also $D_i = C_i$ for $i > 0$ follows because of Lemma 2 in [6] and the injectivity of ϕ on $\mathcal{G}\setminus\mathcal{N}$ is proved.

The formula $F\phi = \phi S$ is an immediate consequence of $FB_{k,l} = B_{k-1,l+1}$.

To prove the last assertion we compute

$$\phi[\dots D_{-1}D_0]_0 = \bigcap_{l=0}^{\infty} B_{0,l} \subseteq \bigcap_{l=0}^{\infty}\bigcap_{i=0}^{l} F^i Z_i$$

where Z_i is the unique element of $\mathcal{W}$ with $D_{-i} \subseteq Z_i$. Hence $\phi[\dots D_{-1}D_0]_0$ is a subset of an element of $\mathcal{I}_\infty$ and with $\Pi\phi[\dots D_{-1}D_0]_0 = \Pi\bigcap_{l=0}^{\infty} B_{0,l} = D_0$ the proof is finished. ∎

PROPOSITION 1: *Suppose T is topologically transitive and $\mathcal{V}$ is a generator. If the family of maps $y \mapsto \frac{\partial g}{\partial y}(x,y)$ is uniformly Hölder continuous and $t \in (0,z)$, then an ergodic F-invariant probability measure μ on X exists with $h_\mu + t\int \varphi\, d\mu > 0$ and $\mu(\bigcup_{I\in\mathcal{I}_{\infty,\sigma}} I) > 0$ for all $\sigma \in E$.*

Proof: Lemma 4 implies the existence of a function $\hat{\psi}$ which is constant on the elements of $\mathcal{Z}_N$ and which fulfills $p(F,\psi)-\varepsilon \le p(F,\hat{\psi})$. Since $\sup_{\mathbf{x}\in\Pi^{-1}P}\psi(\mathbf{x}) < 0$ holds we compute using the functions and definitions of Lemma 4 and its proof

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sup_{\mathbf{x}\in\Pi^{-1}P} S_n\hat{\psi}(\mathbf{x}) &\le \limsup_{n\to\infty}\frac{1}{n}\sup_{\mathbf{x}\in\Pi^{-1}P} S_n\tilde{\psi}(\mathbf{x})\\ &\le \limsup_{n\to\infty}\frac{1}{n}\sup_{\mathbf{x}\in\Pi^{-1}P}(S_n\psi(\mathbf{x}) + 2\|u\|)\\ &= \limsup_{n\to\infty}\frac{1}{n}\sup_{\mathbf{x}\in\Pi^{-1}P} S_n\psi(\mathbf{x}) < 0.\end{aligned}$$

Hence we get

$$\limsup_{n\to\infty}\sqrt[n]{\|e^{S_n\hat{\psi}}\|} = \exp\left(\limsup_{n\to\infty}\frac{1}{n}\sup_{\mathbf{x}\in P} S_n\hat{\psi}(\mathbf{x})\right) < 1 < e^{p(F,\hat{\psi})} = r(\mathbf{M})$$

and we can apply Lemma 6 to get $\mathbf{v} \in l^\infty(\mathcal{D})$ and $\mathbf{u} \in l^1(\mathcal{D})$ with $\mathbf{v}_C > 0$ and $\mathbf{u}_C > 0$ for all $C \in \mathcal{C}$ and $\mathbf{Mv} = r(\mathbf{M})\mathbf{v}$ and $\mathbf{uM} = r(\mathbf{M})\mathbf{u}$. Set $\pi_D = \mathbf{u}_D\mathbf{v}_D$ for $D \in \mathcal{D}$ where one of them is renormed, such that $\sum_{D\in\mathcal{D}}\pi_D = 1$ holds and

$$\mathbf{P}_{DE} = \frac{\mathbf{M}_{DE}\mathbf{v}_E}{r(\mathbf{M})\mathbf{v}_D}.$$

Then $\mathbf{P}$ is a stochastic matrix and $\pi\mathbf{P} = \pi$ holds. We get a Markov measure $\hat{\mu}$ on $\mathcal{G}$.

Because of Lemma 8, (X,F,μ) and $(\mathcal{G},S,\hat{\mu})$ with $\mu = \hat{\mu}\phi^{-1}$ are isomorphic dynamical systems (observe that the set $\mathcal{N}$ in Lemma 8 is a $\hat{\mu}$-nullset). Since $\mathcal{V}$ is a generator for T and hence $\mathcal{Z}$ is a generator for F on X and since entropy is an isomorphism invariant we have $h_\mu(F) = h_{\hat\mu}(S)$. A similar computation to that in [16] page 103 gives $p(F,\hat\psi) = \log r(\mathbf{M}) = h_{\hat\mu}(S) + \int \hat\psi \circ \phi \, d\hat\mu = h_\mu(F) + \int \hat\psi \, d\mu$. Since μ is F-invariant we get $\int \hat\psi \, d\mu \le \int (\psi - u + u \circ F) \, d\mu = \int \psi \, d\mu$ and at last

$$0 < p(F,\hat\psi) = h_\mu(F) + \int \hat\psi \, d\mu \le h_\mu(F) + \int \psi \, d\mu.$$

Choose $\sigma \in E$. Because of the second part of Lemma 5 there is a $D \in \mathcal{C}$ with $\sigma \in D$. Then Lemma 8 and Lemma 6 imply $\mu(\bigcup_{I \in \mathcal{I}_{\infty,\sigma}} I) \ge \hat\mu([D]) = \pi_D > 0$. ∎

We again fix a $t \in (0, z)$ and the measure μ found in the above Proposition.

Lemma 9: *Suppose $\mathcal{V}$ is a generator and the family of maps $y \mapsto \varphi(x,y)$ is uniformly equicontinuous. Set*

$$G = \{\mathbf{x} \in X \colon \lim_{n\to\infty} -\frac{1}{n} \log \mu(I_n(\mathbf{x})) = h_\mu(F) \textit{ and } \lim_{n\to\infty} \frac{1}{n} \log |I_n(\mathbf{x})_{\Pi\mathbf{x}}| = \int \varphi \, d\mu\}.$$

Then $\mu(G) = 1$.

Proof: The mean value theorem and the chain rule imply

$$\log |I_n(\mathbf{x})_x| = \sum_{i=1}^{n} \log \left| \frac{\partial g}{\partial y}(F^{-i}(x,\eta)) \right|$$

for an $\eta \in I_n(\mathbf{x})_x$ and $x = \Pi\mathbf{x}$. For $1 \le i \le n$ we observe that $F^{-i}(\mathbf{x})$ and $F^{-i}(x,\eta)$ are both elements of $F^{-i} I_n(\mathbf{x})_{\Pi F^{-i}\mathbf{x}} = I_{n-i}(F^{-i}\mathbf{x})_{\Pi F^{-i}\mathbf{x}}$. Choose $\varepsilon > 0$ and remember that $|I_{n-i}(F^{-i}\mathbf{x})_{\Pi F^{-i}\mathbf{x}}| \le k^{n-i}$. Since the maps $y \mapsto \varphi(x,y)$ are uniformly equicontinuous for each $x \in P$, there is an $n_0 \in \mathbb{N}$ such that $n - i > n_0$ implies

$$|\varphi(F^{-i}(x,\eta)) - \varphi(F^{-i}(\mathbf{x}))| < \varepsilon.$$

Set $C = |\sup\varphi - \inf\varphi|$. Then we get $|\sum_{i=1}^n \varphi(F^{-i}(x,\eta)) - \sum_{i=1}^n \varphi(F^{-i}(\mathbf{x}))| \le Cn_0 + (n - n_0)\varepsilon$ and we compute

$$\frac{1}{n} \log |I_n(\mathbf{x})_{\Pi\mathbf{x}}| \ge \frac{1}{n} \sum_{i=1}^{n} \varphi(F^{-i}(\mathbf{x})) - \frac{Cn_0}{n} - \frac{(n-n_0)\varepsilon}{n},$$

$$\frac{1}{n} \log |I_n(\mathbf{x})_{\Pi\mathbf{x}}| \le \frac{1}{n} \sum_{i=1}^{n} \varphi(F^{-i}(\mathbf{x})) + \frac{Cn_0}{n} + \frac{(n-n_0)\varepsilon}{n}.$$

Taking $\lim_{n\to\infty}$ and using the ergodic theorem for F^{-1} gives

$$\int \varphi \, d\mu - \varepsilon \le \liminf_{n\to\infty} \frac{1}{n} \log |I_n(\mathbf{x})_{\Pi\mathbf{x}}| \le \limsup_{n\to\infty} \frac{1}{n} \log |I_n(\mathbf{x})_{\Pi\mathbf{x}}| \le \int \varphi \, d\mu + \varepsilon$$

for almost all $\mathbf{x}$. Since ε was arbitrary we get $\lim_{n\to\infty} \frac{1}{n}\log|I_n(\mathbf{x})_{\Pi\mathbf{x}}| = \int \varphi\, d\mu$ for almost all $\mathbf{x}$.

For the second part of the proof we use the Shannon–McMillan–Breiman Theorem for F^{-1}. Observe that $\mathcal{I}_n = \{\bigcap_{i=0}^n (F^{-1})^{-i} Z_i \neq \emptyset\colon Z_i \in \mathcal{Z}\}$ holds. Hence we get $\lim_{n\to\infty} -\frac{1}{n}\log\mu(I_n(\mathbf{x})) = h_\mu(F^{-1}) = h_\mu(F)$ for almost all $\mathbf{x}$ since $\mathcal{Z}$ is a generator for F. Collecting these results we get $\mu(G) = 1$. ∎

LEMMA 10: *Define G as in Lemma 9. Then either $I_\infty(\mathbf{x}) \cap G = \emptyset$ or $I_\infty(\mathbf{x}) \cap G = I_\infty(\mathbf{x})$ for all $\mathbf{x} \in X$.*

Proof: Suppose $\mathbf{x} \in I_\infty(\mathbf{x}) \cap G$. Choose a $\mathbf{y} \in I_\infty(\mathbf{x})$. We have $I_n(\mathbf{y}) = I_n(\mathbf{x})$ for all $n \geq 0$. Hence $\lim_{n\to\infty} -\frac{1}{n}\log\mu(I_n(\mathbf{y})) = h_\mu(F)$ follows.

Choose $\varepsilon > 0$. Since the maps $y \mapsto \varphi(x, y)$ are uniformly equicontinuous for each $x \in P$ and because of the second part of Lemma 3 there is an $n_0 \in \mathbb{N}$ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| < \varepsilon$ holds for $\mathbf{x}$ and $\mathbf{y}$ in the same element of $\mathcal{Z}_{n_0,n_0}$. Since $F^{-i}(\mathbf{x})$ and $F^{-i}(\mathbf{y})$ are elements of the same $\mathcal{Z}_{i,i}$ for $i \geq 0$ we get by a similar computation as in the proof of the above lemma even with the same constant C

$$\frac{1}{n}\log|I_n(\mathbf{y})_{\Pi\mathbf{y}}| \geq \frac{1}{n}\sum_{i=1}^n \varphi(F^{-i}(\mathbf{x})) - \frac{2Cn_0}{n} - \frac{(n-2n_0)\varepsilon}{n},$$
$$\frac{1}{n}\log|I_n(\mathbf{y})_{\Pi\mathbf{y}}| \leq \frac{1}{n}\sum_{i=1}^n \varphi(F^{-i}(\mathbf{x})) + \frac{2Cn_0}{n} + \frac{(n-2n_0)\varepsilon}{n}.$$

Taking $\lim_{n\to\infty}$ finishes the proof since $\varepsilon > 0$ was arbitrary. ∎

For a subset S of X_σ let $\mathcal{U}(\varepsilon, \mathcal{I}_\sigma, S)$ be the set of all finite or countable covers of S by sets of the form $I \cap \Pi^{-1}\sigma$ for $I \in \mathcal{I}_\sigma$ with diameter less than or equal to ε. Define

$$\nu_{t,\mathcal{I}_\sigma}(S) = \lim_{\varepsilon\to 0} \inf_{C\in\mathcal{U}(\varepsilon,\mathcal{I}_\sigma,S)} \sum_{I\in C} |I|^t \quad \text{and} \quad HD_{\mathcal{I}_\sigma}(S) = \inf\{t\colon \nu_{t,\mathcal{I}_\sigma}(S) = 0\}.$$

This is well-defined since $\lim_{n\to\infty} |I_n(\mathbf{x})_{\Pi\mathbf{x}}| = 0$. A similar definition was first used in [9]. We will use the following property of $HD_{\mathcal{I}_\sigma}$ which is also proved there: If $S_k \subseteq X_\sigma$ for $k \geq 0$ and $S = \bigcup_{k\geq 0} S_k$ then $HD_{\mathcal{I}_\sigma}(S) = \sup_{k\geq 0} HD_{\mathcal{I}_\sigma}(S_k)$. Furthermore the usual Hausdorff dimension and $HD_{\mathcal{I}_\sigma}$ do not differ on subsets of G_σ which is shown by the following lemma.

LEMMA 11: *Let $S \subseteq G_\sigma$. Then $HD(S) \geq HD_{\mathcal{I}_\sigma}(S)$.*

Proof: Set $\int \varphi\, d\mu = \Phi$ and fix $\varepsilon > 0$ such that $\Phi + 2\varepsilon < 0$ still holds (remember $\int \varphi\, d\mu < 0$). Similarly as in the proof of Lemma 9 we get for $y \in S$

$$|I_n(\sigma, y)_\sigma| \geq \exp\left(\sum_{i=1}^{n} \varphi(F^{-i}(\sigma, y)) - Cn_0 - (n - n_0)\varepsilon\right),$$

$$|I_n(\sigma, y)_\sigma| \leq \exp\left(\sum_{i=1}^{n} \varphi(F^{-i}(\sigma, y)) + Cn_0 + (n - n_0)\varepsilon\right).$$

Since $(\sigma, y) \in G$ we find a constant $M(y) > 0$ such that $M(y)^{-1}e^{n(\Phi-2\varepsilon)} \leq |I_n(\sigma, y)_\sigma| \leq M(y)e^{n(\Phi+2\varepsilon)}$ holds. Set $S_m = \{y \in S\colon M(y) \leq m\}$. Then $S_m \subseteq S_{m+1}$ and $\bigcup_{m\geq 1} S_m = S$.

Let $\mathcal{U}_\vartheta$ be a cover of S_m by open intervals with length less than or equal to ϑ. Fix $U \in \mathcal{U}_\vartheta$ and let n be minimal such that $|I_n(\sigma, y)_\sigma| \leq |U|$. Then $|I_{n-1}(\sigma, y)_\sigma| \geq |U|$. With $N = me^{-(\Phi+2\varepsilon)} > 0$ we get $m^{-1}e^{n(\Phi-2\varepsilon)} \leq |U| \leq Ne^{n(\Phi+2\varepsilon)}$ and

$$\frac{|I_{n-1}(\sigma, y)_\sigma|}{|I_n(\sigma, y)_\sigma|} \leq mNe^{4n\varepsilon}.$$

Hence we can compute

$$|I_n(\sigma, y)_\sigma| \geq \frac{1}{mN}e^{-4n\varepsilon}|U| \geq c|U|^{1-4\varepsilon/(\Phi+2\varepsilon)}$$

with $c = \frac{1}{mN}N^{4\varepsilon/(\Phi+2\varepsilon)}$ since $n \leq \frac{1}{\Phi+2\varepsilon}\log\frac{|U|}{N}$.

Two different $I_n(\sigma, y)$ are disjoint or one contains the other. Hence we can find a cover $\mathcal{O}_U$ of $U \cap S_m$ by disjoint $I_n(\sigma, y)$ constructed above. We have $\sum_{I\in\mathcal{O}_U} |I| \leq 3|U|$ since $|I| \leq |U|$ for all $I \in \mathcal{O}_U$. We get

$$card(\mathcal{O}_U) \leq 3|U|\frac{1}{c}|U|^{-1+4\varepsilon/(\Phi+2\varepsilon)} = \frac{3}{c}|U|^{4\varepsilon/(\Phi+2\varepsilon)}$$

and

$$\sum_{I\in\mathcal{O}_U} |I|^t \leq \frac{3}{c}|U|^{t+4\varepsilon/(\Phi+2\varepsilon)}.$$

With this estimate for all $U \in \mathcal{U}$ we get for $\mathcal{O} = \bigcup_{U\in\mathcal{U}} \mathcal{O}_U$

$$\sum_{I\in\mathcal{O}} |I|^t \leq \frac{3}{c}\sum_{U\in\mathcal{U}} |U|^{t+4\varepsilon/(\Phi+2\varepsilon)} \quad \text{and} \quad \nu_{t,\mathcal{I}_\sigma}(S_m) \leq \frac{3}{c}\nu_{t+4\varepsilon/(\Phi+2\varepsilon)}(S_m).$$

Set $C(\varepsilon) = 3/c$ and $D(\varepsilon) = 4\varepsilon/(\Phi + 2\varepsilon)$. Then $C(\varepsilon) > 0$ and $D(\varepsilon) < 0$ and $\lim_{\varepsilon\to 0} D(\varepsilon) = 0$. Then $HD(S_m) \geq HD_{\mathcal{I}_\sigma}(S_m)$ follows. Otherwise one could find $t < HD_{\mathcal{I}_\sigma}(S_m)$ and $\varepsilon > 0$ such that $HD(S_m) < t + D(\varepsilon) < HD_{\mathcal{I}_\sigma}(S_m)$ holds. Then $\nu_{t,\mathcal{I}_\sigma}(S_m) \leq C(\varepsilon)\nu_{t+D(\varepsilon)}(S_m) = 0$ would follow, a contradiction.

Since $m \in \mathbb{N}$ was arbitrary the desired result follows. ∎

We now prove that z is also a lower bound for $HD(X_\sigma)$. With Lemma 2 this gives the

Proof of Theorem 1: It remains to prove $z \le HD(X_\sigma)$ for $\sigma \in E$ if $z > 0$. So choose an arbitrary $t \in (0, z)$ and let μ be the measure constructed in Proposition 1 with $h_\mu + t\int \varphi\, d\mu > 0$ and $\mu(\bigcup_{I\in\mathcal{I}_{\infty,\sigma}} I) > 0$ for $\sigma \in P$. This implies $t \le h_\mu(F)/(-\int \varphi\, d\mu)$ since $\int \varphi\, d\mu < 0$. Let G be defined as in Lemma 9. With Lemma 11 we get $HD_{\mathcal{I}_\sigma}(G_\sigma) \le HD(G_\sigma) \le HD(X_\sigma)$ for $\sigma \in E$. Hence the missing step is to prove $h_\mu(F)/(-\int \varphi\, d\mu) \le HD_{\mathcal{I}_\sigma}(G_\sigma)$.

To do this observe that we have

$$\lim_{n\to\infty} \frac{\log \mu(I_n(\sigma,y))}{\log |I_n(\sigma,y)_\sigma|} = \frac{h_\mu(F)}{-\int \varphi\, d\mu} \quad \text{for } (\sigma,y) \in G_\sigma.$$

Furthermore $\bigcup_{I\in\mathcal{I}_{\infty,\sigma}} I \cap G = \bigcup_{y\in G_\sigma} I_\infty(\sigma,y)$ because of Lemma 10 and $\mu(\bigcup_{I\in\mathcal{I}_{\infty,\sigma}} I \cap G) = \mu(\bigcup_{I\in\mathcal{I}_{\infty,\sigma}} I) > 0$ since G has measure 1.

Choose $\vartheta > 0$ arbitrary and define

$$S_k = \left\{ y \in G_\sigma \colon \frac{h_\mu(F)}{-\int \varphi\, d\mu} - \vartheta \le \frac{\log \mu(I_n(\sigma,y))}{\log |I_n(\sigma,y)_\sigma|} \text{ for all } n \ge k \right\}.$$

Then $S_k \subseteq S_{k+1}$ and $G_\sigma = \bigcup_{k\ge 1} S_k$ holds. Hence we get

$$\lim_{k\to\infty} \mu\Big(\bigcup_{y\in S_k} I_\infty(\sigma,y)\Big) = \mu\Big(\bigcup_{y\in G_\sigma} I_\infty(\sigma,y)\Big) > 0.$$

For all $n \ge k$ we have

$$(|I_n(\sigma,y)|_\sigma)^{\frac{h_\mu(F)}{-\int \varphi\, d\mu} - \vartheta} \ge \mu(I_n(\sigma,y)).$$

Choose k_0 large enough, such that $\mu(\bigcup_{y\in S_{k_0}} I_\infty(\sigma,y)) > 0$ holds. For all $\varepsilon \le \min_{y\in S_{k_0}} |I_{k_0}(\sigma,y)_\sigma|$ we get for $\mathcal{B} \in \mathcal{U}(\varepsilon, \mathcal{I}_\sigma, S_{k_0})$

$$\sum_{I\in\mathcal{B}} |I|^{\frac{h_\mu(F)}{-\int \varphi\, d\mu} - \vartheta} \ge \sum_{\substack{I\in\mathcal{I}_\infty \\ I_\sigma \in \mathcal{B}}} \mu(I) \ge \mu\Big(\bigcup_{y\in S_{k_0}} I_\infty(\sigma,y)\Big) > 0$$

and

$$HD_{\mathcal{I}_\sigma}(S_{k_0}) \ge \frac{h_\mu(F)}{-\int \varphi\, d\mu} - \vartheta$$

follows. Since ϑ was arbitrary and $S_k \subseteq G_\sigma$ we get

$$HD_{\mathcal{I}_\sigma}(G_\sigma) \ge \frac{h_\mu(F)}{-\int \varphi\, d\mu}. \quad ∎$$

4. The formula for $HD(X)$

We give the proof of Theorem 2 here. There is a general theory for any Borel subset of $\mathbb{R}^n$, which connects the Hausdorff dimension of the whole set and the Hausdorff dimension of vertical intersections. We refer to Corollary 7.12 in [1] to obtain $HD(X) \geq z+1$. The reverse inequality needs a different, but similar calculation as in the proof of Theorem 1. Furthermore we need some information about the geometric form of the $I \in \mathcal{I}_n$ for an $n \geq 0$.

LEMMA 12: *Suppose that the assumptions of Theorem 2 are fulfilled. For $I \in \mathcal{I}_n$ and $n \geq 0$ set $d(I) = \sup_{\sigma \in \Pi I} |I_\sigma|$. Then there is an integer A and a covering $\mathcal{U}(I)$ by squares with sidelength $d(I)$ such that $card(\mathcal{U}(I)) \leq A/d(I)$ holds for each I.*

Proof: Choose an arbitrary $I \in \mathcal{I}_n$ for an $n \geq 0$. Lemma 5 in [15] shows that the boundary of I consists of two vertical lines and two differentiable functions. Denote the supremum of the absolute value of the slope of these two functions by $s(I)$. Set $\inf_{x\in[0,1]} |T'x| = E > 1$ and remember that we have set $k := \sup|\partial g/\partial y|$. Then the proof of Proposition 2 in [15] shows that there is a $C < \infty$ such that

$$s(I) \leq C \sum_{i=1}^{n+1} \left(\frac{k}{E}\right)^i.$$

Since $k/E < 1$ we get

$$s(I) \leq C \sum_{i=0}^{n+1} \left(\frac{k}{E}\right)^i \leq \frac{C}{1-\frac{k}{E}} < \infty.$$

Let J be an interval in $[0,1]$ with length $d(I)$. Then $I \cap \Pi^{-1}J$ is contained in a rectangle $J \times K$ with $|K| \leq d(I) + 2|J|S(I)$. Let A be the least integer greater than $1 + 2S(I)$. Then $I \cap \Pi^{-1}J$ can be covered by A squares with sidelength $d(I)$. Since $|\Pi I| \leq 1$ we get that I can be covered by $A\frac{1}{d(I)}$ squares of sidelength $d(I)$ and the proof is finished since A is independent of the choice of I. ∎

Proof of Theorem 2: For $I_n \in \mathcal{I}_n$ set $Z_n = F^{-n} I_n$ and $\nu = \left(T^n|_{\Pi Z_n}\right)^{-1}(\sigma)$. Then the chain rule gives

$$|I_{n\sigma}| \leq \sup_{\mathbf{x}\in\Pi^{-1}\nu} \prod_{i=0}^{n-1} \left|\frac{\partial g}{\partial y}(F^i\mathbf{x})\right|.$$

We furthermore get

$$d(I_n) = \sup_{\sigma\in\Pi I_n} |I_{n\sigma}| \leq \sup_{\mathbf{x}\in Z_n} \prod_{i=0}^{n-1} \left|\frac{\partial g}{\partial y}(F^i\mathbf{x})\right|$$

since $\sigma \in \Pi I_n$ is equivalent to $\nu \in \Pi Z_n$.

Fix $\vartheta \in (0,1)$ arbitrarily. Then there is an n_0 such that $d(I_n) \leq \vartheta/\sqrt{2}$ for all $I_n \in \mathcal{I}_n$ with $n \geq n_0$, since

$$\left|\frac{\partial g}{\partial y}(\mathbf{x})\right| \leq k < 1 \quad \text{for all } \mathbf{x}.$$

Of course $\bigcup_{I_n \in \mathcal{I}_n} I_n$ covers X. For $I_n \in \mathcal{I}_n$ let $\mathcal{U}(I_n)$ be the cover of I_n found in the above Lemma 12. Then $\bigcup_{I_n \in \mathcal{I}_n} \mathcal{U}(I_n)$ is a cover of X by subsets with diameter less than or equal to ϑ.

Choose $t > z+1$. Then we get using Lemma 12

$$\begin{aligned}
\nu_{t,\vartheta}(X) &\leq \sum_{I_n \in \mathcal{I}_n} \sum_{U \in \mathcal{U}(I_n)} (d(I_n))^t \sqrt{2}^t \\
&\leq \sqrt{2}^t \sum_{I_n \in \mathcal{I}_n} \frac{A}{d(I_n)} d(I_n)^t \\
&\leq A\sqrt{2}^t \sum_{Z_n \in \mathcal{Z}_n} \exp\left((t-1) \sup_{\mathbf{x} \in Z_n} \sum_{i=0}^{n-1} \log\left|\frac{\partial g}{\partial y}(F^i \mathbf{x})\right|\right) \\
&= A\sqrt{2}^t p_n(F, (t-1)\varphi)
\end{aligned}$$

for all $n \geq n_0$. Since $t-1 > z$ we have $p_n(F,(t-1)\varphi) < 1$ for large n. Therefore $\nu_{t,\vartheta}(X) \leq A\sqrt{2}^t$. Hence $\nu_t(X) < \infty$ holds for all $t > z+1$ and $HD(X) \leq z+1$ follows. ∎

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